1/ Math 112: Introductory Real Analysis

§ The Riemann-Stieltjes integral

Def Let $[a,b]$ be a given interval.

A partition P of $[a,b]$ is a finite set of points $x_0, x_1, \cdots, x_n$

where $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.

We write $\Delta x_i := x_i - x_{i-1}$ $(i = 1, \cdots, n)$.

Suppose f is a bounded real function on $[a,b]$.

Corresponding to each partition P of $[a,b]$, we put

$$M_i := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

$$m_i := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

$$U(P,f) := \sum_{i=1}^{n} M_i \Delta x_i \quad \text{(upper sum)},$$

$$L(P,f) := \sum_{i=1}^{n} m_i \Delta x_i \quad \text{(lower sum)}.$$

Define the upper Riemann integral to be

$$\overline{\int_a^b} f\,dx := \inf\{U(P,f) \mid P \text{ is a partition of } [a,b]\},$$

and the lower Riemann integral to be

$$\underline{\int_a^b} f\,dx := \sup\{L(P,f) \mid P \text{ is a partition of } [a,b]\}.$$

If $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$, we say f is Riemann integrable

and denote the common value by $\int_a^b f\,dx$, the Riemann integral.

Rmk The upper and lower integrals are defined for every bounded function f.

This is because $m \leq f(x) \leq M \quad (a \leq x \leq b)$ implies

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a),$$

for any partition P of $[a,b]$.

Def Let α be a monotonically increasing function on $[a,b]$.

For each partition $P = \{a = x_0, \cdots, x_n = b\}$ of $[a,b]$,

We write $\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1})$

For any bounded real function f on $[a,b]$, we put

$$U(P,f,\alpha) := \sum_{i=1}^{n} M_i \Delta\alpha_i,$$

$$L(P,f,\alpha) := \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

Define $\overline{\int_a^b} f\,d\alpha := \inf_P U(P,f,\alpha)$,

$\underline{\int_a^b} f\,d\alpha := \sup_P L(P,f,\alpha)$.

If they are equal, we say f is integrable with respect to α

and call the common value $\int_a^b f\,d\alpha$ the Riemann-Stieltjes integral of f with respect to α.

Def We say that a partition P^* is a refinement of P if $P^* \supseteq P$.

Given two partitions P_1 and P_2, $P^* = P_1 \cup P_2$ is their common refinement.

Thm If P^* is a refinement of P, then

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P,f,\alpha).$$

proof) By induction, it suffices to consider the case when P^* contains just one more point than P. Let this extra point be $x^* \in (x_{i-1}, x_i)$.

Put $m_i^{(1)} := \inf\limits_{x \in [x_{i-1}, x^*]} f(x)$ and $m_i^{(2)} := \inf\limits_{x \in [x^*, x_i]} f(x)$.

Clearly $m_i^{(1)} \geq m_i$ and $m_i^{(2)} \geq m_i$.

Hence $L(P^*,f,\alpha) - L(P,f,\alpha)$

$$= m_i^{(1)}\left(\alpha(x^*) - \alpha(x_{i-1})\right) + m_i^{(2)}\left(\alpha(x_i) - \alpha(x^*)\right) - m_i\left(\alpha(x_i) - \alpha(x_{i-1})\right)$$

$$= \left(m_i^{(1)} - m_i\right)\left(\alpha(x^*) - \alpha(x_{i-1})\right) + \left(m_i^{(2)} - m_i\right)\left(\alpha(x_i) - \alpha(x^*)\right) \geq 0.$$

The proof of $U(P^*,f,\alpha) \leq U(P,f,\alpha)$ is analogous. ∎

Thm $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$.

proof) For any two partitions P_1 and P_2,

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha),$$

where $P^* := P_1 \cup P_2$ is the common refinement.

Taking sup over P_1 and inf over P_2, we get the desired inequality. ∎

<u>Thm</u> (Criterion for integrability) Let f be a bounded real function on $[a,b]$.
Then f is integrable with respect to α if and only if
for every $\varepsilon>0$ there exists a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon.$$

proof) $(\Rightarrow)$ Suppose f is integrable, and let $\varepsilon>0$ be given.
Then there exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha)-\int f\,d\alpha<\frac{\varepsilon}{2},$$

$$\int f\,d\alpha-L(P_1,f,\alpha)<\frac{\varepsilon}{2}.$$

Let $P=P_1\cup P_2$. Then

$$U(P,f,\alpha)\leq U(P_2,f,\alpha)<\int f\,d\alpha+\frac{\varepsilon}{2}<L(P_1,f,\alpha)+\varepsilon\leq L(P,f,\alpha)+\varepsilon.$$

$(\Leftarrow)$ For every $\varepsilon>0$, there is P such that $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$,

so $0\leq\overline{\int}f\,d\alpha-\underline{\int}f\,d\alpha\leq U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$.

Since this is true for every $\varepsilon>0$, we have $\overline{\int}f\,d\alpha=\underline{\int}f\,d\alpha$. ∎

<u>Thm</u> If f is continuous on $[a,b]$, then it is integrable (with respect to α).

proof) Let $\varepsilon>0$ be given. Choose $\eta>0$ so that $(\alpha(b)-\alpha(a))\eta<\varepsilon$.
Since f is uniformly continuous on $[a,b]$, there exists $\delta>0$ such that

$$|f(x)-f(y)|<\eta \quad \text{whenever} \quad x,y\in[a,b] \text{ and } |x-y|<\delta.$$

If P is any partition of $[a,b]$ such that $\Delta x_i<\delta$ for all i,

$$U(P,f,\alpha)-L(P,f,\alpha)=\sum_{i=1}^{n}(M_i-m_i)\Delta\alpha_i\leq\sum_{i=1}^{n}\eta\,\Delta\alpha_i$$
$$=\eta(\alpha(b)-\alpha(a))<\varepsilon.$$

Therefore, f is integrable. ∎